INVESTIGATING THE IMPORTANCE OF CHOOSING A CORRECT PRIOR DISTRIBUTION IN A BAYESIAN APPROACH TO INVENTORY MANAGEMENT

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ABSTRACT

Bayesian approaches have been introduced and recommended in the inventory management literature. The objective of these formulations is to obtain an estimate of the future demand rate of products. The Bayesian treatment typically involves a prior distribution about a parameter of distribution of demand. In this study, a simulation is conducted to explore the behavior of a proposed approach when the distribution of the parameters is different from that of prior distribution used in the mathematical development of the procedure. This study uses a distribution for the demand rate to be a mixture of two Gamma distributions. The simulation study reveals that this mixture can make the Bayesian approach to be no better than an approach in which no prior is assumed. The Bayesian approach is compared to a procedure in which the predicted demand rate of a product is estimated using only historical demand on that product without assuming any distribution about the demand rates of a pool of products. This study differs from previous simulation studies investigating Bayesian approaches in that this simulation investigates the robustness of using a gamma prior distribution when a mixture of two gamma distribution is the proper prior for the demand rates of a pool of products.

USING BAYESIAN APPORACHES TO ESTIMATING FUTURE INVENTORY LEVELS

A number of studies have illustrated the usefulness of a Bayesian approach in inventory management (Silver, 1965; De Wit, 1983; Price and Haynsworth, 1986; and Aronis, Magou, Dekker, and Tagaras, 2004). The importance of predicting future sales, sometimes with little historical data is presented in forecasting models by Willemain, Smart, Shockor, & DeSautels (1994), Johnston & Boylan (1996), Syntetos & Boylan (2001). The objective of Bayesian approaches to inventory management is to optimize holding inventory by forming a distribution of the demand rate. Reducing total cost due to shortages and surpluses of goods can be critical to a business' strategic plan to become more efficient (Hollier, Mak, & Lai, 2002; Razi & Tarn, 2003). Popovic (1987) proposes a Bayesian approach to inventory decision making, which allows the estimates of the parameters of the a priori distribution of demand rate λ to be updated providing the foundation for this paper.

A Proposed Bayesian Model for Inventory Decision Making

The Bayesian approach is different from the traditional Poisson approach in that it uses a prior distribution for demand rate λ. Popovic (1987) proposes Bayesian approaches to inventory decision making related to uncertain demand rates. Since the rate λ is not known, an *a priori* distribution of λ is assigned a gamma distribution Γ(α, ß) with the probability density function:

$$f(\lambda) = \frac{\beta^{\alpha}\lambda^{\alpha-1}e^{-\beta\lambda}}{\Gamma(\alpha)}. \qquad (1)$$

The unconditional distribution of demand is then:

$$P\{X_t = k\} = \int_0^{\infty}\frac{(\lambda t)^k}{k!}e^{-\lambda t}f(\lambda)d\lambda = \binom{\alpha+k-1}{k}\left[\frac{\beta}{(\beta+t)}\right]^{\alpha}\left[\frac{t}{(\beta+t)}\right]^{k} \qquad (2)$$

where $k = 0, 1\ldots$ and $\binom{n}{x}$ is the number of combinations of taking *x* things from *n* distinct things.

Thus X_t has a negative binomial distribution NB(α, ß/(ß + *t*)) when $P\{X_t = k\}$ is denoted by p_k:

$$p_k = [\frac{(\alpha+k-1)}{k(\beta+1)}]p_{k-1}, p_0 = \left[\frac{\beta}{\beta+1}\right]^{\alpha} \qquad (4)$$

By applying Bayes' Rule,

$$f(\lambda \mid X_1) = \frac{[P(X|\lambda)f(\lambda)\}}{\int_0^{\infty}P(X|\lambda)f(\lambda)d\lambda,} \qquad (3)$$

and the posteriori distribution of λ over the first time interval I_1 is then

$$f(\lambda \mid X_1) = \frac{(\beta+1)^{\alpha+X_1}\lambda^{(\alpha+X_1-1)}e^{-(\beta+1)\lambda}}{\Gamma(\alpha+X_1)} \qquad (4)$$

Assume each time period has a length of one. The distribution of demand for period 2 is

$$P\{X_{I_2} = k\} = \left[\int_0^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}f(\lambda \mid X_1)d\lambda\right] = \left[\frac{\Gamma(\alpha+k-1)}{k!}\Gamma(\alpha+X_1)\right]\left[\frac{\beta+1}{(\beta+2)}\right]^{\alpha+X_1}\left[\frac{1}{(\beta+2)}\right]^{k}, \qquad (5)$$

Thus $X_{I_2} \sim$ NB(α + X_1, $\frac{(ß+1)}{(ß+2)}$). It then follows that the *a posteriori* distribution of λ for the second time interval I_2 if demand X_2 occurs will be

$$\lambda | X_1, X_2 \sim \Gamma(\alpha + X_1 + X_2, \beta + 2) \text{ or generally } \lambda | X_1, X_2, \ldots, X_n \sim \Gamma(\alpha + \sum_{i=1}^{n} X_i, \beta + n) \qquad (6)$$

and the distribution of demand at time interval I_{n+1} is:

$$X_{I_{n+1}} \sim \text{NB}\left(\alpha + \sum_{i=1}^{n} X_i, \left(\frac{\beta+n}{\beta+n+1}\right)\right) \qquad (7)$$

Determining Optimal Inventory Levels Based on Costs and Distribution

The optimal inventory level in the second unit time interval I_2 can be computed by considering the cost per unit time of a surplus C_1 and a shortage C_2 of an item as well as the demand described by the *a posteriori* distribution of demand (Popovic, 1987). As additional demand information is accumulated, the *a posteriori* distribution of demand can be updated to improve the accuracy of the estimate. Depending on the surplus cost and shortage cost of a single item, an inventory level, r_1^*, is determined at the beginning of the first time interval to minimize the cost. The optimal value of r will make the following inequality true.

$$\sum_{k=0}^{r_1^*-1}\binom{\alpha+k-1}{k}\left(\frac{\beta}{\beta+1}\right)^{\alpha}\left(\frac{1}{\beta+1}\right)^{k} < \frac{C_{shortage}}{C_{surplus}+C_{shortage}} < \sum_{k=0}^{r_1^*}\binom{\alpha+k-1}{k}\left(\frac{\beta}{\beta+1}\right)^{\alpha}\left(\frac{1}{\beta+1}\right)^{k} \quad (10)$$

Furthermore, Popovic (1987) shows that the Poisson distribution can be used in place of the Negative Binomial distribution for the number of demands in the time period under study, providing the basis for the Poisson model used in this paper. For each simulation a true alpha and beta are selected to define two gamma distributions that represent the true demand rate for each product in one of two groups. Shortage costs and surplus costs are provided for the given distribution. The ratio of surplus to shortage is the critical factor in the model. These four parameters are varied in a series of simulations to explore the behavior of the Bayes and Poisson models presented in this paper for a group of 100 products split into two groups that varies in size for each figure.

Three sets of data points are constructed for each set of parameters given in the simulation. For demand following a combination of two given gamma distributions, each defined by an alpha, the shape parameter, and beta, the scale parameter, costs are generated dependent on shortage and surplus costs for a group of 100 products for 10 future periods. A demand rate is randomly selected for each product. Once this demand rate λ is selected, "actual" product sales are generated using a Poisson process as is common in the literature. Estimates of the demand rate are made by the Bayes and Poisson models after observing the observed sales for each period. From these estimates the amount of inventory to hold for the next time interval is computed. Expected costs from holding this amount of inventory is then determined using the theoretical underlying distribution of demand for that product. Three costs are compared for each product: 1. Bayes model cost, 2. Poisson model costs, and 3. Optimal costs based on the Theoretical demand rates.

SIMULATION RESULTS

The results of the simulations conducted in this research are shown in the following 14 graphs. The simulations represented in each graph, generate total inventory costs for 100 products, divided into two groups, using two models with the theoretical cost of demand known. The actual demand rate for each product in the group is generated using the gamma distribution with two parameters, alpha and beta.

Each simulation investigates the effect on holding and shortage costs of the 100 products. Each group of 100 products consists of two lots of products. One lot has a mean demand rate of 3 (Group 1) and the other lot has a mean demand rate of 5 (Group 2). The group with the mean of 3 has a variance of 3 representing a more homogeneous group than the second lot with a mean of 5. The lot of products with a mean of 5 has a variance of 500, suggesting that despite having a mean of 5, the lot of products is varied. For the group of 100 products the number of products is varied in each lot in subsequent simulations. In simulation 1, lot 1 (μ=3) has no products and lot 2 (μ=5) has 100 products. In simulation 3, lot 1 (μ=3) has 100 products and lot 2 (μ=5) has 0 products. Simulation 2 is 50/50.

Further more, two different cost ratios are investigated. The first group of simulations (Figures 1 to 3) have a 1 to 1 ratio of surplus to shortage costs. That is the surplus cost is the same as the shortage costs. For the second group of simulations (Figures 4 to 6) the ratio of surplus to shortage costs is 1 to 5, so if extra inventory costs $10 running out of will cost $50.

The first group of simulations (Figure 1 to 3) with equal shortage and surplus costs, see a gradual decrease in costs as the number of products in the lot with a mean of 5 and high variance is reduced. The Bayes sees the greatest advantage over the Poisson model when ever all 100 products in the group have the same mean and variance. For mixed lots, the Bayes advantage is marginal over the Poisson model. After ten simulations both models approach the theoretical costs, with neither model having a distinct advantage. The second group of simulations shows a gradual decrease in costs as the number of products in the lot with a mean of 5 and a high variance is reduced. The Bayes outperforms the Poisson model in each simulation of between about 4 and 20%. After ten simulations both models approach the theoretical costs.

Simulation Graphs for Equal Costs for Surplus and Shortage (Figures 1 to 3)

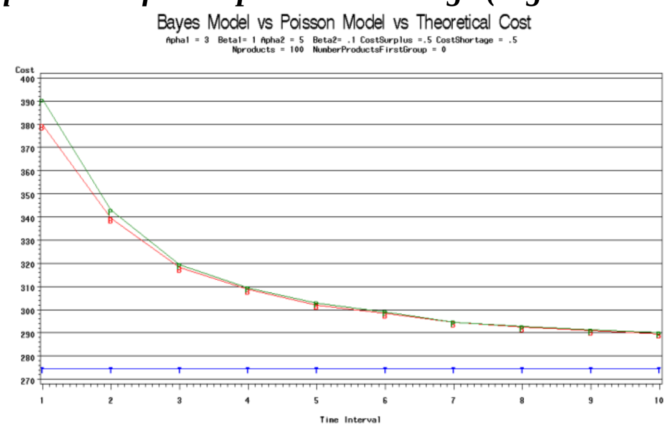

Figure 1. Inventory Cost Assuming Equal Costs for Surplus and Shortage of Products and No Products Belonging to Group 1.

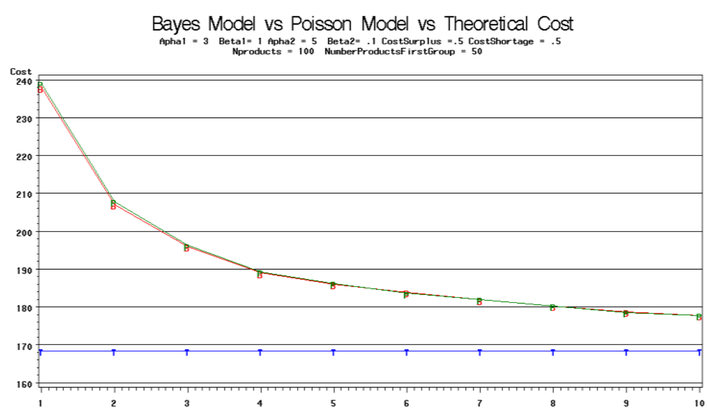

Figure 2. Inventory Cost Assuming Equal Costs for Surplus and Shortage of Products and 50% of Products Belonging to Group 1.

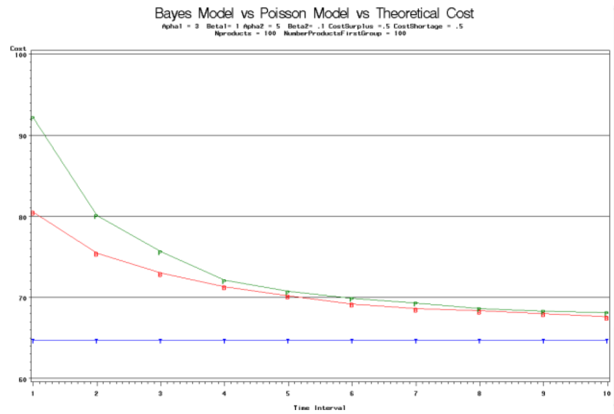

Figure 3. Inventory Cost Assuming Equal Costs for Surplus and Shortage of Products and 100% of Products Belonging to Group 1.

Simulation Graphs for Unequal Costs for Surplus and Shortage (Figures 4 to 6)

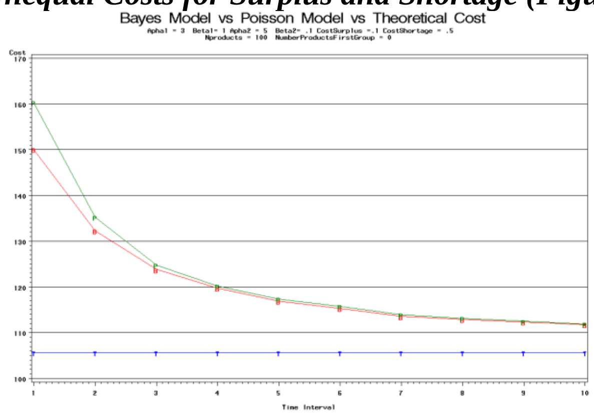

Figure 4. Inventory Cost Assuming Unequal Costs for Surplus and Shortage of Products and No Products Belonging to Group 1.

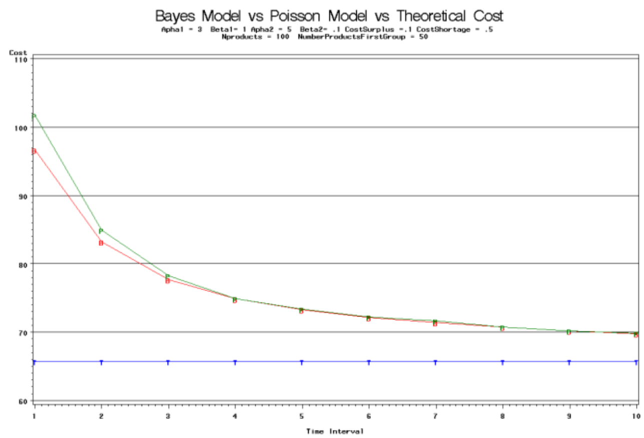

Figure 5. Inventory Cost Assuming Unequal Costs for Surplus and Shortage of Products and 50% of the Products Belonging to Group 1.

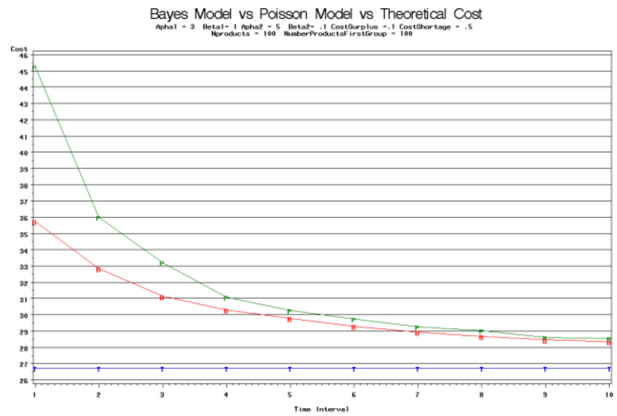

Figure 6. Inventory Cost Assuming Unequal Costs for Surplus and Shortage of Products and 100% of the Products Belonging to Group 1.

CONCLUSIONS

The results of this study are different from previous simulations in that a prior distribution that is a gamma is used to approximate the mixture of two gammas. All graphs show that as the number of time units increases both the Poisson and Bayes model approach the theoretical values, since better estimates can be produced as historical information is gathered. Furthermore, the graphs suggest that the behavior of the actual data being forecasted will determine which model works best when forecasting sales. The comparisons of the simulations in group 1 to group 2 are the most striking. When the surplus and shortage costs are similar, even for lots with different mean demands and variances, the advantage of the Bayes model over the Poisson model is marginal at best. However, when the surplus and shortage costs are not similar, that is one is much more than the other, the Bayes model provides estimates that are closer to the theoretical values of between 4% and 20% depending on the lot mix.

The proposed models are appropriate for new products with very little or no demand history to help determine stocking decisions at the introductory point of the product. Additional simulations are required to determine which range of demand rates is appropriate for each model. Very high or very low demand rates could alter the proposed selection guidance.

Limitations

Many variables will affect the outcome of the simulations. The model predicts one demand rate for all items that have yet to experience customer demand. Demand for each item is assumed to be independent. However since family of items are used to forecast demand rate, this might not always be the case. It is possible that usage of some items could be correlated to other items and would not be independent. The number of items within a family with and without sales will affect the accuracy of the model. Furthermore, the model assumes that the demand rate for all items in each lot comes from the same distribution (gamma distribution with the same parameters). If the actual population demand rate changes for items in the same family, the model will be affected. External factors could change the demand rates as well. The model assumes that parts are not subject to external factors.

REFERENCES

Aronis, K.-P., Magou, I., Dekker, R., & Tagaras, G. (2004). Inventory control of spare parts using a Bayesian approach: A case study. *European Journal of Operational Research, 154*(3), 730-739.

De Wit, J. R. (1983). Inventory problems with slow moving items: A Bayesian approach. The Statistician, 32(1/2), 201-206.

Hollier, R. H., Mak, K. L., & Lai, K. K. (2002). Computing optimal (*s, S*) policies for inventory systems with a cut-off transaction size and the option of joint replenishment. *International Journal of Production Research, 40*(14), 3375-3389.

Johnston, F. R., & Boylan, J. E. (1996). Forecasting for items with intermittent demand. *Journal of the Operational Research Society, 47*(1), 113-121.

Popovic, J. B. (1987). Decision-making on stock levels in cases of uncertain demand rate. *European Journal of Operational Research, 32*(2), 276-290.

Price, B. A., & Haynsworth, H. C. (1986). How to prepare inventory forecasts for very low demand items. *The Journal of Business Forecasting, 5*(2), 21-22.

Razi, M. A., & Tarn, J. M. (2003). An applied model for improving inventory management in ERP systems. *Logistics Information Management, 16*(2), 114-124.

Silver, E. A. (1965). Bayesian determination of the reorder point of a slow moving item. *Operations Research, 13*(6), 989-997.

Syntetos, A. A., & Boylan, J. E. (2001). On the bias of intermittent demand estimates. *International Journal of Production Economics, 71*(1-3), 457-466.

Waller, M., Johnson, M. E., & Davis, T. (1999). Vendor managed inventory in the retail supply chain. *Journal of Business Logistics, 20*(1), 183-203.

Willemain, T. R., Smart, C. N., Shockor, J. H., & DeSautels, P. A. (1994). Forecasting intermittent demand in manufacturing: A comparative evaluation of Croston's method. *International Journal of Forecasting, 10*, 529-538.